Multicriteria Decision Analysis in Water Resources Management

Edited by:

Janos J. Bogardi

Hans-Peter Nachtnebel

A compendium based on the short intensive course on "Decision Support Techniques for Integrated Water Resources Management". Organized by the International Training Centre (PHLO) and the Department of Water Resources of the Wageningen Agricultural University, The Netherlands, held from 10-15 June 1991 in Wageningen. (Contribution of The Netherlands to the fourth phase of the International Hydrological Programme of UNESCO).

SC.94/WS.14

6 MULTICRITERION DECISION MAKING METHODS WITH ORDINAL AND CARDINAL SCALES: ELECTRE I-III

H.P. NACHTNEBEL

ABSTRACT Often, the outcomes of alternatives cannot be quantitatively assessed. However, the decision maker is able to judge the outcomes qualitatively such as good, poor and bad. In this case a distance based technique cannot be applied because even in the case that one alternative is obviously better than any other it cannot be quantified how much it is better.

A class of techniques called ELECTRE is explained here which is able to handle also ordinally expressed criteria. In its simplest form ELECTRE I, only a partial ranking of the alternatives can be obtained and it has also to be considered that this procedure yields sometimes intransitive outrankings.

ELECTRE II and III result in a complete ranking of the alternatives but require additional information about the preference structure of the decision maker. The techniques can be applied only in the case of a discrete set of alternatives.

6.1 INTRODUCTION

In this chapter a class of outranking techniques called ELECTRE (ELimination Et Choix Traduissant la REalité) which was developed by Benayoun et al. (1966) and Roy et al. (1968, 1971, 1973, 1975, 1978), will be presented. This methodology handles quantitative, qualitative and fuzzy criteria and by pairwise comparison of alternatives a ranking procedure is provided. The method can only be applied to a discrete set of alternatives. First, an outline of the assumption and the methodology is presented and then some applications are given for ELECTRE I, II, III.

6.2 METHODOLOGY

A multicriterion problem can be formally described by a set of alternatives, criteria and preference values. A finite set of alternatives $A = \{a_i\}$ $(i = 1, m)$ has to be evaluated. It is assumed that each alternative is completely identified by a set of a criteria $Z = \{z_j\}$ $(j = 1, n)$.

An evaluation procedure, which is discussed later in detail, aggregates the criteria into a single index, for instance a ranking index to assist in:

- the selection of the "best" alternative a_i^*
- rejecting the "bad" ones a_b^- or identifying the "good" ones a_g^+
- the complete ordering of the alternatives.

In **simple deterministic outranking** procedures a comparison is expressed by a binary relation indicating that a pairwise comparison can result either in *'one alternative is obviously better than the other'* or that *'it is not the case'* and one obtains evaluations such as:

- a_i preferred to a_j
- a_i indifferent to a_k (6.1)
- a_j incomparable with a_k.

The first case holds when the outcome f_{il} of alternative a_i with respect to criterion z_l is preferred or at least equal to f_{jl} for all criteria. In the second case a_i is preferred to a_l only in some of the criteria while in some others a_j is more attractive. The introduction of weights, associated to every criterion might result in a reduced set of indifferent alternatives.

The latter case may occur when a pairwise comparison of alternatives does not provide reliable or enough information to the decision maker. A graphical representation of a pairwise comparison is given in restricted form in Fig. 6.1. By definition, an arc is directed from the preferred alternative to the dominated alternative. According to a pairwise comparison of alternatives the following preference relations are obtained:

$a_2 > a_1$		
$a_1 > a_4$	$a_1 > a_5$	
$a_4 > a_3$	$a_5 > a_3$	
$a_4 > a_6$	$a_5 > a_8$	$a_8 > a_7$
$a_4 > a_7$	$a_5 > a_7$	

Because of a simple pairwise comparison it cannot be concluded from $a_2 > a_1$ and $a_1 > a_4$ that $a_2 > a_4$, where $>$ means 'preferred to' or 'better than'.

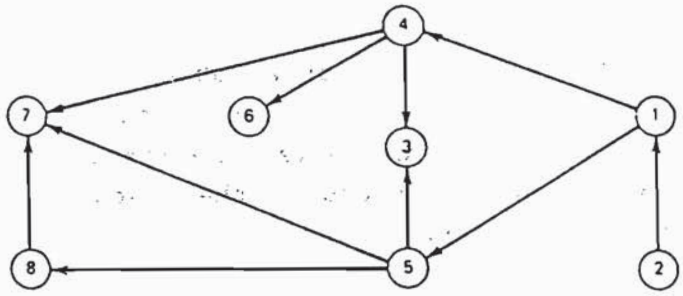

FIG. 6.1 Simplified graph representing selected pairwise comparison

In other words, a pairwise comparison might also imply results such as

- a_i preferred to a_j
- a_j preferred to a_k (6.2)
- a_k preferred to a_i.

Often, it is not obvious that alternative a_i is preferred to a_j but there is a high possibility that a_i is better than a_j. Thus, an extension of the deterministic outranking is in the fuzzy outranking. A **fuzzy outranking** relation assigns "credibility", expressed by a fuzzy set, to any comparison. The evaluation in equ. (6.1) may be expressed by:

comparison	credibility
$(a_i > a_j)$	0.6
$(a_j > a_i)$	0.1
$(a_i > a_k)$	0.8
$(a_k > a_i)$	0.7
$(a_j > a_k)$	0.2
$(a_k > a_j)$	0.5

High values of credibility correspond to preference or indifference while low values are assigned to incompatible or clearly dominated alternatives. A membership grade or credibility of 0.9 that a_i is better than a_j indicates a high possibility, where 1 expresses that a_i is definitely better than a_j. More details are given in chapter 16.

In specific cases some conclusion can be drawn without any computation from the alternatives versus criteria array (pay-off table). For instance, an alternative is for sure better than another if it dominates with respect to all criteria.

6.3 ELECTRE I

The goal of ELECTRE I is to select a subset of alternatives which are preferred for most of the criteria and which do not fail drastically with respect to one or more criteria. The first attribute is expressed by the concordance and the second by the discordance index. These two indices refer to the performance of alternatives with respect to criteria.

The output matrix or pay-off matrix F with elements f_{ik} contains the output of alternative a_i with respect to criterion z_k and is given in any units. Subsequently it is assumed that a higher output value is preferred to a lower output.

6.3.1 Concordance

A weight w_k is assigned to each criterion z_k expressing the importance of a specific criterion in comparison to other criteria. Often, criteria can be grouped with respect to several subgoals such as the "preservation of environmental quality" or the "improvement of regional economy". In these cases the sum of weights of criteria associated to one subgoal expresses the preference structure and this should be considered in the phase of weighting each criterion. Otherwise a bias might be introduced by weighting.

The concordance C(i,j) between any two alternatives i and j is a weighted measure of the number of criteria for which alternative i is preferred to or at least as good as alternative j and is given by:

$$C(i,j) = \frac{\text{sum of weights for criteria where } i \geq j}{\text{total sum of weights}} \qquad (6.3)$$

Concordance can be thought of as the weighted percentage of criteria for which one alternative is preferred to another.

Mathematically, we calculate on the basis of a pairwise comparison the sum of weights where $a_i > a_j$ for the criterion z_k

$$W^+ = \sum w_k, \quad \text{for all } k\text{: } f_{ik}>f_{jk} \qquad (6.4)$$

In the opposite case we obtain:

$$W^- = \sum w_k, \quad \text{for all } k\text{: } f_{ik}<f_{jk} \qquad (6.5)$$

and for indifferent outcomes:

$$W^{=} = \sum w_k. \quad \text{for all } k: f_{ik}=f_{jk} \tag{6.6}$$

$$C(i,j) = \frac{W^{+} + 1/2\ W^{=}}{W^{+} + W^{=} + W^{-}} \tag{6.7}$$

Summarizing, the concordance index is based on an ordinal comparison neglecting the degree of a_i being better than a_j.

6.3.2 Discordance

To compute the discordance an interval scale is first defined for each criterion. The scale V_{Ok} is used to compare the discomfort caused between the "worst" and the "best" of each criterion. Each criterion can be assigned a different range V_{Ok}. The output f_{ik}, expressed in any unit, is mapped on the scale V_{Ok} by:

$$V_{ik} = \left| \frac{f^{-}_{k} - f_{ik}}{f^{*}_{k} - f^{-}_{k}} \right| \cdot V_{Ok} \tag{6.8}$$

f^{*}_{k} stands for the best and f^{-}_{k} for the worst possible result with respect to criterion k.

In equ. (6.8) a simple linear mapping function was used which implies that an increase in the output of, say 10%, is appreciated in the same way, independently from the level of performance. Given this information, the discordance index D(i,j) expressing the discomfort of preferring a_i compared to a_j is defined as:

$$D(i,j) = \frac{\text{maximum interval where } j>i}{\text{total range of scale}} \tag{6.9}$$

$$\text{or } D(i,j) = \text{Max } (f_{jk} - f_{ik})/k^{*} \tag{6.10}$$

$$k^{*} = \text{Max } (V_{Ok}) \qquad \{k: k = 1,n\}$$

6.3.3 Ranking

To synthesize both the concordance and discordance matrices, threshold values p, q between (0,1) are defined. In choosing a value of p, the decision maker specifies the amount of concordance that is required for further considerations. In specifying q, he specifies the amount of discordance he is willing to tolerate. The result of ELECTRE I is a preference graph representing a partial ordering of the alternative systems.

A graph is formed in which each node represents a nondominated solution. The alternatives which are obtained on the basis of defined levels of p and q define a subset B. This subset satisfies at least once the condition:

$$C(i,\ell)>p \quad \text{for } \ell = 1,m$$
$$D(i,\ell)<q \quad \text{for } \ell = 1,m \tag{6.11}$$

In Fig. 6.2 an example of a typical graph is given. The arrows emanating from the nodes are called directed paths and correspond to the pairwise comparison of alternatives. The statement "i preferred to j" corresponds to a path starting from i in direction of j. This methodology selects a subset of nondominated alternatives and by varying p and q the number of elements in B can be controlled.

The alternatives which are not included in B are not considered as being attractive solutions. Either they fail in an unacceptable degree with respect to one or more criteria or their overall performance is poor.

The selection procedure is performed in several steps and is described below. In the first step the indices C (i,j) and D (i,j) are calculated according to (6.7) and (6.9).

TABLE 6.1 Concordance and discordance matrices

C =

-	0.3	0.4	0.45	0.45	0.4	0.5
0.7	-	0.6	0.4	0.4	0.35	0.45
0.6	0.4	-	0.3	0.3	0.4	0.5
0.55	0.6	0.7	-	0.5	0.6	0.4
0.55	0.6	0.7	0.5	-	0.7	0.45
0.6	0.65	0.6	0.4	0.3	-	0.35
0.5	0.55	0.5	0.6	0.55	0.65	-

$$D = \begin{bmatrix} - & 0.2 & 0.2 & 0.4 & 0.4 & 0.4 & 0.8 \\ 0.25 & - & 0.25 & 0.25 & 0.2 & 0.25 & 0.6 \\ 0.2 & 0.25 & - & 0.2 & 0.25 & 0.2 & 0.6 \\ 0.2 & 0.25 & 0.2 & - & 0.25 & 0.2 & 0.4 \\ 0.25 & 0.2 & 0.25 & 0.25 & - & 0.25 & 0.4 \\ 0.4 & 0.4 & 0.2 & 0.2 & 0.25 & - & 0.4 \\ 0.4 & 0.4 & 0.25 & 0.25 & 0.2 & 0.25 & - \end{bmatrix}$$

In the next step the DM specifies a minimum concordance condition of 0.6 and a maximum discordance condition of 0.20; that is $C(i, j) \geq 0.6$ and $D(i, j) \leq 0.20$. With this specification the graph G_c can now be constructed. The directed paths which appear in the graph are determined by the set of indices that simultaneously satisfy the requirement that $p \geq 0.6$ and $q \leq 0.20$. These indices are:

(3,1), (4,3), (4,6), (5,2), (6,3) and (7,6)

where, for example, (3,1) means that alternative a_3 is better than a_1.

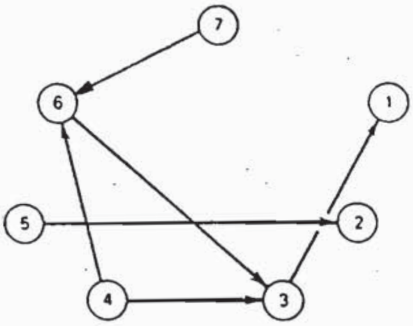

FIG. 6.2 Typical graphs for ELECTRE I for $p \geq 0.6$ and $q \leq 0.20$ and matrices in table 6.1.

The alternatives included in the set B exhibit different ranks. for instance, alternative 3 is not a good solution because 4 and 6 are better candidates. Due to the lack of graphs between 7 and 4, and between 5 and 4 we can conclude that alternatives 7, 4, 5 are more attractive than other ones.

Roy et al. (1968) formalized this approach in defining a kernel K from the set B which satisfies the following conditions:

* there is no pairwise preference among the elements in set K
* all the elements outside of K are dominated by at least one of the elements in K

In our example this yield the following elements for K.

7, 5, 4 and surprisingly 1

Obviously, these are no pairwise preference relations among these elements and thus the first conditions is satisfied. And all the other alternatives {2,3,6} are dominated by at least one of the elements of K. It is somewhat counter-intuitive that alternative 1 is within the kernel because it is worse than 3 but 1 is not dominated by 4 and is therefore included in the kernel. If someone would try to replace 1 by 3 as an element in the kernel he would fail because 4 would dominate 3 and therefore the first condition would be violated.

6.4 ELECTRE II

As it has been previously stated ELECTRE I does not necessarily provide a transitive and complete ranking of the alternatives.

ELECTRE II is an extension of ELECTRE I and was developed by Roy (1968, 1971, 1974, 1975) and Roy and Bertier (1971) and offers a complete ordering of the nondominated alternatives. This ordering is accomplished by the construction of outranking relationships based on the predefined preferences of the DM.

As in the case of ELECTRE I, alternative i is preferred to alternative j (i.e., i outranks j) if and only if the concordance and the discordance condition are both satisfied. Additionally, multiple levels of concordance and discordance are specified and are used to construct two extreme outranking relationship; that is, a strong relationship R_s and a weak relationship R_w which will be briefly described below.

6.4.1 Concordance

Concordance and discordance definitions of ELECTRE II differ slightly from those of ELECTRE I. Let W^+, $W^=$ and W^- be defined as in the case of ELECTRE I. That is, W^+ represents the sum of the weights for which alternative i is preferred to alternative j, $W^=$ represents the sum of the weights for which alternative i is indifferent to alternative j, and W^- represents the sum of weights for which alternative j is preferred to alternative i. With this observation, the concordance index for ELECTRE II is defined as:

$$C(i,j) = \frac{W^+ + W^=}{W^+ + W^= + W^-} \tag{6.12}$$

The concordance test is a necessary condition for alternative i to outrank alternative j, and is expressed by the following inequalities:

$$C(i,j) > p$$

and (6.13)

$$W^+ > W^-$$

where $0 < p < 1$ is the required minimum level of concordance. In the concordance matrix those values which also satisfy the condition $W^+ > W^-$ will be marked with a superscript asterisk (*).

6.4.2 Discordance

Again, the discordance index is obtained by mapping the outcome f_{ik} of alternative a_i with respect to criterion z_k in an interval scale. This procedure provides the opportunity to compare criteria expressed in different units and is here formulated in a more general form than previously. The scaled cardinal difference of two alternatives i and j with respect to criterion k is:

$$d'_{ijk} = (f_{jk} - f_{ik})/F(V_{Ok},S(k)) \quad \text{for all } f_{ik} < f_{jk} \qquad (6.14)$$

and the worst is the discordance measure

$$d(i,j) = \text{Max } (d'_{ijk}) \qquad (6.15)$$

V_{Ok} is the scale of criterion k and S(k) controls the importance of the difference in the respective outcome. Often, the function F is assumed to be the same for all criteria and is given by:

$$F(V_{Ok}, S(k)) = \text{Max } (V_{Ok}) = k^* \quad \{k:k = 1,m\} \qquad (6.16)$$

The general formulation in (6.14) provides the opportunity to assess discordances differently for each criterion and it could be also used to consider the level of performance in the discordance index. In other words the outcomes of two alternatives achieving their goals (criteria) with 80% and 90% would yield a different discordance index than in the case where their performance is 20% and 30%. The denominator in (6.14) would be equal in these cases.

Applying eq. (6.16) the discordance index is obtained in the same way as in ELECTRE I.

6.4.3 Strong and weak outranking relationship

Essentially, ELECTRE II establishes a complete ordering on the set of alternatives being considered such that it satisfies:

1. The test of concordance (i.e., the concordance measure is above some minimum level of acceptability).
2. The test of nonconcordance (i.e. discordance measure is below some maximum level of allowable discordance).

ELECTRE II requires two preference graphs representing the strong and weak preferences of the decision maker. The strong preference graph results from the use of stringent threshold values; that is, the decision maker is asked to select a high level of concordance and a low level of discordance. For the weak preference graph, the decision maker is asked to relax his threshold values (lower p, higher q). These relaxed threshold values can be viewed as lower bounds on the system performance that the decision maker is willing to accept. The algorithm for ordering the systems from preference graphs is presented by Abi Ghanem et al. (1978).

The outranking procedure consists of constructing two relationships:

- a strong relationship, R_s,
- and weak relationship, R_w.

When R_s is used, a better discrimination is obtained than using R_w.

In order to define R_s and R_w, let p^*, p^0, and p^- represent three decreasing levels of concordance such that $0 < p^- < p^0 < p^* < 1$. Furthermore, let q^0 and q^* represent two increasing levels of discordance such that $0 < q^0 < q^* < 1$. With these specifications R_s is defined if and only if one (or both) of the following sets of conditions hold:

$$\text{I.}\quad \begin{aligned} &C(i,j) > p^* \\ &D(i,j) < q^* \\ &W^+ > W^- \end{aligned} \tag{6.17}$$

or

$$\text{II.}\quad \begin{aligned} &C(i,j) > p^0 \\ &D(i,j) < q^0 \\ &W^+ > W^- \end{aligned} \tag{6.18}$$

If at least one set of conditions holds, then alternative a_i strongly outranks alternative a_j. The weak relationship R_w, is defined if and only following conditions hold:

$$C(i,j) > p^-$$
$$D(i,j) < q^* \qquad (6.19)$$
$$W^+ > W^-$$

Then alternative a_i is said to outrank weakly alternative a_j.

As a result of the two pairwise relationships, two graphs can be constructed similar to ELECTRE I; one for the strong relationship and one for the weak relationship. These graphs are then used in an iterative procedure to obtain the desired ranking of the alternatives (Fig. 6.3).

The final ranking procedure is achieved in a three-stage process. First, a downward ranking R is obtained. Next, an upward ranking, R' is made. Then the final ranking which is called the median ranking, R_o, is achieved which is a blend of the downward and upward rankings. It is worth to note that here downward and upward does not directly correspond to R_s and R_w. Furthermore, both R_s and R_w are used to obtain R and R' and finally R_o. Sometimes, downward and upward is replaced by strong and weak ranking but it should not be confused with R_s and R_w.

Downward ranking corresponds to select the best, the second best etc. while the upward ranking tries to find the worst, the second worst alternative in an ascending order.

A subset G_s of A which was obtained by the strong relationship R_s provides the basis by the downward (strong) ranking procedure. Eight steps described below have to be taken.

In this subsection the index k refers to an iteration index and is not related to a criterion z_k. Let Y(k) be a subset of G_s where Y(0) = G_s. The set of the best alternatives A(k) which will receive the ranks k + 1 is chosen by using the following algorithm:

1. Set k = 0.
2. Select all nodes of Y(k) not having a precedent (i.e. the alternatives not being outranked by other elements). Let H represent this set.
3. Identify all nodes in H that are related through R_w as depicted in G_w. Let U represent this set.
4. Select all nodes in U not having a precedent in G_w. Denote this set as B.
5. Define A(k) as

$$A(k) = (H - U) \cup B$$

where H - U is the relative complement of U with respect to U, that is, $H - U = \{x : x \in H, x \notin U\}$.

6. Obtain a ranking for every $x \in A(k)$ by setting R(x) = k + 1.
7. Set Y (k + 1) = Y(k) - A(k).
8. If Y(k + 1) is the null set, stop. Otherwise set k = k + 1 and

return to step two.

The upward (weak-)ranking algorithm embodies the above algorithm and consists of three steps:

1. Reverse the directions of the arcs in G_s and G_w.
2. Obtain a ranking, a(x), for each alternative x as was done in the strong ranking algorithm (substituting a(x) for R(x) in step 6).
3. Readjust the ranking process by setting:

$$R'(x) = 1 + a_{max} - a(x), \tag{6.20}$$

where X is the set of all nondominated alternatives and

$$a_{max} = \max_{x \in X} a(x)$$

The reason is that in the upward ranking procedure the weakest alternative is identified first and attains the rank 1, and the second worst the rank 2, etc. To achieve consistency with the downward ranking equ. (6.20) is applied, changing only the sequence in ranking. An example is given in Fig. 6.4 and in Tables 6.2 to 6.4.

The final ranking R_0 is obtained by locating R_0 between R and R'. One way, suggested by Roy and Bertier (1971) is to define:

$$m(x) = \frac{R + R'}{2} \tag{6.21}$$

and then rank in a decreasing manner the values given by the averaging function m(x). This processes yields the final ranking $R_0(x)$.

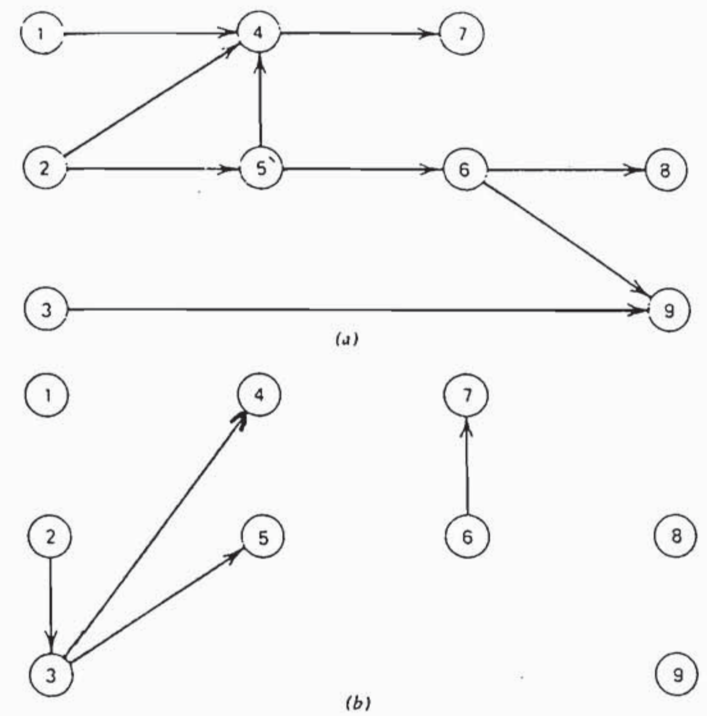

FIG. 6.3 Downward ranking
Strong and weak relationships
(a). Reduced graph of the strong relationship
(b). Reduced graph of the weak relationship.

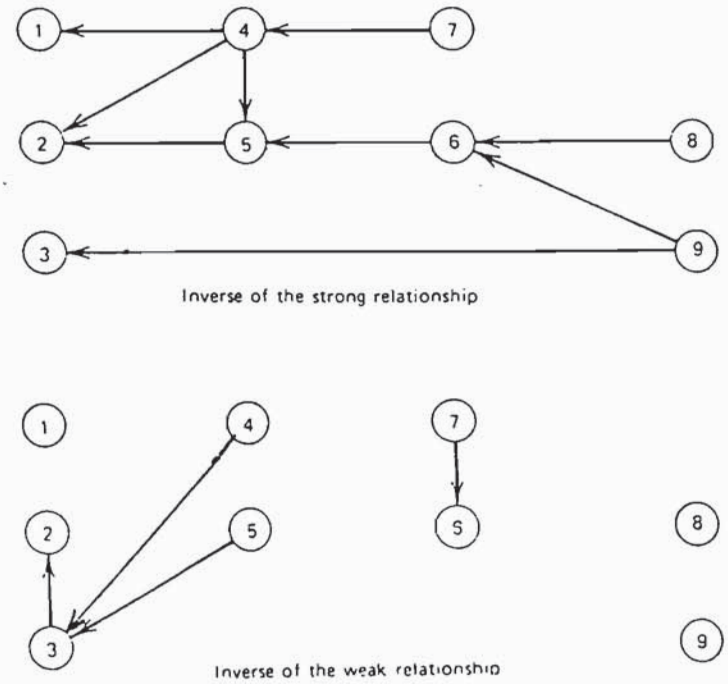

FIG. 6.4 Upward ranking
(a) Inverse of the strong relationship
(b) Inverse of the weak relationship

TABLE 6.2 Downward ranking procedure

Steps	Result	Alternative	Rank
1	K = 0	a_i	R
2	H {1,2,3}		
3	U {2,3}		
4	B {2}		
5	A {1,2}		
6	K = 1	1	1
7	Y {3,4,5,6,7,8,9}	2	1
2	H {3,5}		
3	U {3,5}		
4	B {3}		
5	A {3}		
6	K = 2	3	2
7	Y {4,5,6,7,8,9}		
2	H {5}		
3	V (ø}		
4	B {ø}		
5	A {5}		
6	K = 3	5	3
7	Y {4,6,7,8,9}		
2	H {4,6}		
3	U {ø}		
4	B {ø}		
5	A {4,6}	4	4
6	K = 4	6	4
7	Y {7,8,9}		
2	H {7,8,9}		
3	U {ø}		
4	B {ø}		
5	A {7,8,9}	7	5
6	K = 5	8	5
7	Y {ø}	9	5

TABLE 6.3 Upward ranking procedure

Steps	Result	Alternative	Rank	
1	K = 0	a_i	a(x)	R'
2	H {7,8,9}			
3	U {ø}			
4	B {ø}			
5	A {7,8,9}	7	1	5
6	K = 1	8	1	5
7	Y {1,2,3,4,5,6}	9	1	5
2	H {3,4,6}			
3	U (3,4}			
4	B {4}			
5	A {4,6}	4	2	4
6	K = 2	6	2	4
7	Y {1,2,3,5}			
2	H {1,3,5}			
3	U {3,5}			
4	B {5}			
5	A {1,5}	1	3	3
6	K = 3	5	3	3
7	Y {2,3}			
2	H {2,3}			
3	U {2,3}			
4	B {3}			
5	A {3}			
6	K = 4	3	4	2
7	Y {2}			
2	H {2}			
3	U {ø}			
4	B {ø}			
5	A {2}			
6	K = 5	2	5	1
7	Y {ø}			

TABLE 6.4 Final ranking

Alternatives	1	2	3	4	5	6	7	8	9
Downward Rank R	1	1	2	4	3	4	5	5	5
Upward Rank R'	3	1	2	4	3	4	5	5	5
Median Rank R_0	2	1	2	4	3	4	5	5	5

6.5 ELECTRE III

The main goal of all the ELECTRE methods is to obtain a preference relation for all alternatives. This preference $r(i,j)$ expresses the degree that alternative a_i is preferred to a_j. $r(i,j) = 1$ indicates that a_i is at least as good as a_j.

The preference relation is derived from the concordance index $C(i,j)$ and the corresponding discordance index $D(i,j)$. This approach implies among other aspects that

$$r(i,j) - r(j,i) \neq 0 \tag{6.22}$$

Introducing a preference threshold level r_0 an alternative a_i is preferred to a_j if:

$$r(i,j) - r(j,i) > r_0. \tag{6.23}$$

It is also possible to define instead of r_0 a function which is dependent on the level of $r(j,i)$ because a difference of 0.2 in the two preference values has a different meaning on different levels of preference.

For the comparison of two alternatives a_i and a_j with respect to the same criterion three threshold levels or functions are defined:

* indifference function I_ℓ
* preference function P_ℓ
* veto function V_ℓ.

Repeating again, the outcome of a_i with respect to z_ℓ is defined as $f_{i\ell}$.

If $f_{j\ell} < f_{i\ell} + I_\ell(f_{i\ell})$ it can be concluded that a_i is at least as good as a_j with respect to z_ℓ.

If $f_{j\ell} > f_{i\ell} + P_\ell(f_{i\ell})$ a_j is better that a_i with respect to z_ℓ.

If $f_{j\ell} > f_{i\ell} + V_l(f_{i\ell})$ then a_j is definitely better than a_i. Even a better outcome of a_i with respect to another criterion z_k cannot compensate for.

It is evident that $V_\ell > P_\ell > I_\ell > 0$ for any level $f_{i\ell}$. Note that P_ℓ refers to the comparison of two alternatives with respect to the criterion z_ℓ while r_{ij} refers to the overall comparison of two alternatives.

6.5.1 Concordance

The concordance index with respect to a single criterion z_l ZC_{ijl} is defined according to:

$$ZC_{ij\ell} = 1 \text{ if } f_{j\ell} < f_{i\ell} + I_\ell\,(f_{i\ell})$$
$$ZC_{ij\ell} = 0 \text{ if } f_{j\ell} > f_{i\ell} + P_\ell\,(f_{i\ell}) \qquad \text{else} \qquad (6.24)$$

$$ZC_{ij\ell} = \frac{f_{j\ell} - f_{i\ell} - P_\ell(f_{i\ell})}{I_\ell(f_{i\ell}) - P_\ell(f_{i\ell})}$$

Defining:

$$u_{ij\ell} = f_{j\ell} - f_{i\ell} \qquad (6.25)$$

the concordance matrix is obtained by:

$$C(i,j) = C_{ij} = \overset{(1)}{\Sigma}\, W_\ell \quad + \quad \overset{(2)}{\Sigma}\, W_\ell * \frac{u_{ij\ell} - P_\ell}{I_\ell - P_\ell} \qquad (6.26)$$

The summation index in the first term is for all ℓ: $u_{ij\ell} < I_\ell$ and in the second ℓ: $I_\ell < u_{ij\ell} < P_\ell$ (Ostanello, 1983).
Obviously for $I_\ell = P_\ell = 0$ only the first to expressions in (6.24) yield reasonable figures and the same results as in ELECTRE I.

6.5.2 Discordance

Replacing I_ℓ by V_ℓ yields the respective discordance index $ZD_{ij\ell}$ for a specific criterion z_ℓ.

The preference relation r_{ij} is obtained from:

$$r_{ij} = c_{ij} \cdot dm_{ij}$$

$$dm_{ij} = \prod_{l} \frac{1 - ZD_{ij\ell}}{1 - c_{ij}} \qquad l: c_{ij} < ZD_{ij\ell} \tag{6.27}$$

6.5.3 Ranking

The preference relation is subjected to a distillation procedure to assign iteratively ranks to the alternatives. An additional threshold level r_0 which was described in (6.23) is required to distinguish among preference figures. The outranking procedure is quite similar to ELECTRE II. Three preference schemes are applied: downward ranking, upward ranking and median ranking, which represents the final order of alternatives.

6.6 EXAMPLE OF ELECTRE III: HYDROPOWER VERSUS ECOLOGY

Problem Description:
The hydropower potential of the Danube is nearly completely utilized along its course in Austria. Downstream of Vienna several proposals for hydropower schemes have been elaborated from which five will be investigated by ELECTRE III. These alternatives are characterized by different locations along the downstream section, by different design criteria such as utilizable head and length of impoundment (Nachtnebel, 1990; Fleischmann, 1986). In this example ELECTRE III is applied to evaluate the alternatives and to contribute in a rational way to end this long debate.

The objectives are in:
* increased utilization of national resources, especially of hydropower potential
* economic utilization to support national economy
* the support of navigation
* preservation of unique wetland habitats along the Danube.

The criteria characterizing the goals above are

1. annual power generation (in GWh)
2. installed capacity (in MW)
3. investment cost (10^9 ATS, 1 US $ = 11,2 ATS)
4. number of locks
5. required area for impoundment dams (hectares)
6. important ecological areas affected by an alternative
7. length of impounded river sections (in km)

The numbering of the alternatives corresponds to the example given in chapter 10.

The figures are given in Table 6.5. To assure that increasing figures express increasing preferences the data were reshaped by introducing some upper limits for the criteria. Thus the data in Table 6.6 indicate for instance in column (5) the area of flood plain forests not required for impoundment dams.

TABLE 6.5 Plan-impact matrix

Alternatives A	Criteria Z						
	(1) (GWh)	(2) (MW)	(3) (10^9 ATS)	(4) (No)	(5) (ha)	(6) (ha)	(7) (km)
2	409	57.5	5.61	2	20	0	0
6	1718	247.5	24.88	4	70	500	47.8
8	2000	300	15.66	3	180	495	47.8
9	2008	327.5	15.90	3	230	490	47.8
10	2075	365	11.40	2	330	780	37.8

TABLE 6.6 Rescaled plan-impact matrix

	Criteria Z						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Upper Limit	-	-	30.10^9 ATS	10	500 ha	800 ha	-
Unit	1000 GWh	100 MW	10.10^9 ATS	No	100 ha	100 ha	100 km
2	0.409	0.575	2.439	8	4.8	8.0	0.0
6	1.718	2.475	0.512	6	4.3	3.0	0.478
8	2.000	3.000	1.434	7	3.2	3.05	0.478
9	2.008	3.275	1.410	7	2.7	3.1	0.478
10	2.075	3.650	1.860	8	1.7	0.2	0.378

First, the set of threshold levels given in Table 6.7 was applied. The indifference and the preference levels are 0 and the veto level was set such to avoid any veto condition. This simplifies the approach to the ELECTRE I method.

TABLE 6.7 Threshold levels and weights

	Criteria Z						
	1	2	3	4	5	6	7
Weights	1	1	2	1	1	1	1
Indifference	0	0	0	0	0	0	0
Preference	0	0	0	0	0	0	0
Veto level	2	4	2	3	3.5	8.0	.5

Based on equ. (6.27) the preference matrix expressing the results of a pairwise comparison of alternatives with respect to a criterion is obtained (Table 6.8).

TABLE 6.8 Preference relations $r_0 = 0$.

Alternative	At least as good as alternative no				
	2	6	8	9	10
2	1.0000	0.0676	0.0400	0.0340	0.1117
6	0.0071	1.0000	0.3234	0.1633	0.1043
8	0.1582	0.8750	1.0000	0.6250	0.3750
9	0.1156	0.8750	0.6250	1.0000	0.3750
10	0.0057	0.4286	0.6250	0.6250	1.0000

Assuming a threshold level $r_0 = 0$ the ranking including upward, downward and median ranking is given in Table 6.9. The applied procedure is the same as in ELECTRE II.

TABLE 6.9 Ranking results with $r_0 = 0$

	Alternatives				
	2	6	8	9	10
Downward	4	5	1	1	3
Upward	1	5	3	3	2
Median rank	2.5	5.	2.	2.	2.5

Subsequently, an increased threshold level is assumed ($r = 0.0605$). This level corresponds to the smallest positive difference between two preference indices namely $r_{2,6} - r_{6,2} = 0.0676 - 0.0071 = 0.0605$.

TABLE 6.10 Ranking results obtained with $r_0 = 0.0605$

	Alternatives				
	2	6	8	9	10
Downward	4	4	1	1	3
Upward	1	5	3	3	1
Median rank	2.5	4.5	2.	2.	2.0

A further increase of $r_0 = 0.118$ results in an improvement of the rank of alternative 10. Especially in the upward ranking it achieves the best position.

TABLE 6.11 Ranking results obtained with $r_0 = 0.250$

	Alternatives				
	2	6	8	9	10
Downward	4	4	1	1	3
Upward	1	5	1	1	1
Median rank	2.5	4.5	1.0	1.0	2.0

The effect of changing the weights is given in Table 6.12, where the weight of the costs is decreased and some environmental indicators are increased. In other words, more emphasis is on environmental aspects.

TABLE 6.12 Assignment of new weights

	Criteria Z						
	1	2	3	4	5	6	7
Weight	1	1	1	2.5	2.5	1	1

With a $r_0 = 0.$ the preference relation is given in Table 6.13

TABLE 6.13 Preference relation matrix

Alternative	At least as good as alternative no 2	6	8	9	10
2	1.0000	0.1027	0.0700	0.0686	0.2443
6	0.0041	1.0000	0.6000	0.2967	0.4075
8	0.0858	0.7500	1.0000	0.5500	0.6000
9	0.0627	0.7500	0.6500	1.0000	0.6000
10	0.0032	0.1714	0.3810	0.4000	1.0000

The corresponding ranking with $r_0 = 0$ is given in Table 6.14

TABLE 6.14 Ranking results with $r_0 = 0$

	Alternatives 2	6	8	9	10
Downward	4	3	2	1	5
Upward	1	5	4	2	3
Median rank	2.5	4.	3.	1.5	4.

Increasing the threshold level to $r_0 = 0.0059$ the ranks given in Table 6.15 are obtained.

TABLE 6.15 Ranking with the weights in 6.12 and $r_0 = 0.0059$

	Alternatives 2	6	8	9	10
Downward	4	3	2	1	5
Upward	1	5	4	1	3
Median rank	2.5	4.0	3.0	1.0	4.0

Even an increase in r_0 up to 0.0986 does not change the median ranking sequence.

TABLE 6.16 Ranking with the weights in 6.12 and r = 0.2190

	Alternatives				
	2	6	8	9	10
Downward	2	2	4	1	4
Upward	1	5	1	1	4
Median rank	1.5	3.5	2.5	1.0	4.0

Comparing the ranking obtained with the two sets (Table 6.7 and 6.12) the following conclusions can be drawn:

1. In the first case alternative 10 is always better than alternative 9.
2. In the second case with emphasized environmental objectives alternative 9 becomes more attractive and should be preferred to alternative 10 which is now on position 4 (median rank). The sensitivity with respect to changes in the levels I, P, V has been also investigated in detail.

6.7 CONCLUSIONS

Comparing the three procedures it can be concluded that they are quite similar in their structure. All of them try to establish a preference matrix which is dependent on both the concordance and the discordance index.

Both indices are obtained from a pairwise comparison of alternatives and therefore a full ranking of alternatives is difficult to achieve. In ELECTRE I only a partial ordering of alternatives is obtained. But by repeating the method with different threshold levels for concordance and discordance a preferred set of alternatives can be identified. This method is quite simple and provides a good overview over the possible rankes of alternatives.

ELECTRE II is based on strong and weak outranking relationship by introducing two sets of threshold levels. The method provides a ranking of all alternatives. ELECTRE III utilizes a fuzzy outranking approach by introducing an indifference, a preference and veto level.

The concordance and discordance indices are similarly derived as in the two previously discussed approaches. Additionally a preference index r combining the concordance and discordance matrices is used to achieve a full ranking of alternatives.

With respect to the application of the methods some simple examples were given.

Summarizing, ELECTRE is one of the few techniques which is able to handle both cardinal and ordinal scales. In principal, the method could be also used when differences in the performance of two alternatives are dependent on the performance level itself.

The methodology can only be applied in case of discrete alternatives and it is recommendable to use this technique in the case of a few but quite different alternatives.

Numerous application of the method are found in the literature and also attempts have been taken to extend this procedure for group decision making.

REFERENCES

Abi Ghanem, G., Duckstein, L. & Hekman, L. (1978) Multiobjective analysis of a vegetation management problem using ELECTRE II. WP 78-19. Dept. of Systems and Ind. Eng. Univ. of Arizona, Tucson.

Benayoun, R., Roy, B., & Sussmann, B., (1966) ELECTRE, Une methode pour guider le choix en presence de points de vue multiple. Direction Scientifique, Note de Travail, No. 49, SEMA, Paris.

Fleischmann, A. (1986) Auswertung von Variationen für die energiewirtschaftliche Nutzung der Donau östlich von Greifenstein. Diplomarbeit am Inst. für Ökonometrie und Operations Research, TU Wien.

Nachtnebel, H.P. (1990) Comparison of hydropower utilization and environmental impacts along the Danube downstream of Vienna. Ingenieur et architectes suisse, No. 18, 412-415.

Ostanello, A. (1983) Outranking methods, p. 41-60. In Fandel, G. and Spronk, J. (Eds.) Multiple Criteria Decision Methods and Applications. Springer Verlag, Berlin.

Roy, B. (1968) Classement et choix an presence de points de vue multiples; la methode ELECTRE. RIRO 2, 8,57-75.

Roy, B. & Bertier, B. (1971) La methode ELECTRE II: une methode de classement en presence de critères multiples. Note de travail No. 142. Groupe METRA, LAMSADE, Univ. Dauphine, Paris.

Roy, B. (1973) How outranking relation helps multiple criteria decision making. In Cochrane J.L. and Zeleney M. (Eds.) Multi Criterion Decision Making. University of South Carolina Press, SC.

Roy, B. (1974) Critères multiples et modelisation des préférences l'apport des relation de sur classement. Revue d'economie politique, 84, 1,1-44.

Roy, B. (1975) Partial preference analysis and decision aide: the fuzzy outranking concept. In Bell D., R. Keeney and Raiffa, H. (Eds.) Conflicting objectives in desicion making, p. 40-74, Wiley, NY.

Roy, B. (1978) ELECTRE III: un algorithme de classement fonde sur une representation floue de preference en presence de critères multiples. Cahiers Centre Etudes Recherches Operationelle, LAMSADE, Paris, 20, 1, 3-24.